APPROXIMATE THEORIES OF THE SPINNING OF AIRCRAFT

by

G. R. Walsh*

1. *Introduction*

A systematic mathematical study of the equations of motion of the aeroplane was initiated by Brodetsky [1] in 1940. Among the many manoeuvres the considers is the slow spin, for which the develops an approximate theory based on neglecting small terms in the equations of motion. The magnitude of any term is expressed as a power of sin γ, where γ is the appropriate gliding angle. In a first approximation, only terms of greatest magnitude are retained in each equation. The solution so obtained may be improved by proceeding to a second or higher approximation.

Brodetsky's method, expressed in current notation, forms the basis of the present paper. His theory of the slow spin serves as an introduction to an approximate theory of the fast oscillatory spin in which the angular velocity in pitch is assumed small compared with the angular velocities in roll and yaw.

2. *Equations of Motion*

For simplicity, it is assumed throughout this paper that all controls are centralised. The effects of control movements on spin entry and recovery have been investigated in detail with the aid of a digital computer by Scher and others [2].

The equations of motion of a rigid aircraft, referred to principal axes through the centre of gravity, and including all the terms of possible significance in the spin, may be written in non-dimensional form as follows:**

$$\dot{u} - rv + qw = \tfrac{1}{2} v'^2 (C_{L0} \sin\alpha_0 - C_{D0} \cos\alpha_0) + v' x_w \delta w - \bar{k} \sin\theta, \quad (1)$$

$$\dot{v} - pw + ru = v' y_v v + \bar{k} \cos\theta \sin\phi, \quad (2)$$

$$\dot{w} - qu + pv = -\tfrac{1}{2} v'^2 (C_{L0} \cos\alpha_0 + C_{D0} \sin\alpha_0) + v' z_w \delta w + \bar{k} \cos\theta \cos\phi, \quad (3)$$

$$\dot{p} - A'qr = v' \left(\frac{\mu l_v}{i_A} v + \frac{l_p}{i_A} p + \frac{l_r}{i_A} r \right) + \frac{\mu l_{vw}}{i_A} v\, \delta w + \frac{l_{\dot{v}}}{i_A} \dot{v}, \quad (4)$$

$$\dot{q} - B'rp = v' \left(\frac{\mu m_w}{i_B} \delta w + \frac{m_q}{i_B} q \right) + \frac{m_{\dot{w}}}{i_B} \dot{w}, \quad (5)$$

$$\dot{r} - C'pq = v' \left(\frac{\mu n_v}{i_C} v + \frac{n_p}{i_C} p + \frac{n_r}{i_C} r \right) + \frac{n_{\dot{v}}}{i_C} \dot{v}. \quad (6)$$

These equations are obtained from Brodetsky's paper [1], after adding

* University of York, Heslington, York, England

** For Notation, see p. 247.

certain terms and changing the notation. The variable δw, rather than w, is used for convenience in equations (3) and (5).
In general,

$$w = w_o + \delta w,$$

where the suffix 0 refers to the initial conditions. The non-dimensional factors v' and v'^2 in equations (1) - (6) represent changes in the aerodynamic forces and moments due to a change in the resultant airspeed. When discussing aircraft manoeuvres, it is not always possible to equate these factors to unity.

The kinematic equations are

$$p = \dot{\phi} - \dot{\psi} \sin \theta, \tag{7}$$

$$q = \dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi, \tag{8}$$

$$r = -\dot{\theta} \sin \phi + \dot{\psi} \cos \theta \cos \phi. \tag{9}$$

3. *Brodetsky's Theory of the Slow Spin*

In Brodetsky's equations of motion for the slow spin [1], time is discarded as the independent variable, the Eulerian angle ψ being used instead. Here, time is retained as the independent variable, although essentially we follow Brodetsky's method.

Take C_{D0}, assumed to be approximately 0,25, to represent first order smallness (as previously mentioned, Brodetsky uses $\sin \gamma$, the corresponding quantity in his notation). Using the O notation to denote orders of magnitude expressed in powers of C_{D0}, assume

$$0(C_{L0}) = 0, \quad 0(\bar{k}) = 0, \quad 0(A') = 0(B') = 0, \; 0(C') = 1,$$
$$0(\mu) = -2, \quad 0(x_w) = 0(y_v) = 0(z_w) = 0,$$
$$0(l_v/i_A) = 0, \; 0(l_r/i_A) = -1, \quad 0(m_w/i_B) = 0,$$
$$0(m_q/i_B) = 0, \quad 0(n_v/i_c) = 2.$$

At the stall, the terms containing l_p, n_p and n_r are negligible. Also, the terms containing $l_{\dot{v}}$, $m_{\dot{w}}$, $n_{\dot{v}}$ and l_{vw} are ignored. We assume further that

$$0(v) = 0(w) = 0(\delta w) = 1, \quad 0(p) = 0(q) = 0(r) = 0.$$

Then

$$v' \approx u$$

and

$$0(u) = 0.$$

We now retain only the terms of greatest magnitude in each equation. Equations (1), (2), (3) and (6) become, respectively,

$$\dot{u} = -\bar{k} \sin \theta, \tag{10}$$

$$ru = \bar{k} \cos \theta \sin \phi, \tag{11}$$

$$qu = \tfrac{1}{2} u^2 C_{L0} \cos \alpha_o - \bar{k} \cos \theta \cos \phi, \tag{12}$$

$$\dot{r} = 0. \tag{13}$$

In these equations, every term is of order zero; the equations therefore represent a first approximation to the motion. Equations (7) - (13) may now be solved for the variables $u, p, q, r, \theta, \phi, \dot{\psi}$ as functions of time. The initial conditions are

$$u = u_0,\ r = r_0,\ \phi = \phi_0,\ p = q = \theta = 0,\ \dot{\psi} = \dot{\psi}_0.$$

From equation (13),

$$r = r_0$$

and then from equation (11),

$$u = \bar{k} \cos\theta \sin\phi / r_0. \tag{14}$$

Substituting the initial conditions in (12) gives

$$\bar{k} = u_0^2 C_{L0} \cos\alpha_0 / 2\cos\phi_0 \tag{15}$$

From equations (8) and (9)

$$\dot{\psi} = q \sin\phi \sec\theta + r \cos\phi \sec\theta,$$

and using equations (12) and (11), this gives

$$u\dot{\psi} = \left(\frac{1}{2} u^2 C_{L0} \cos\alpha_0 - \bar{k}\cos\theta\cos\phi\right)\sin\psi \sec\theta + \bar{k}\sin\phi\cos\phi.$$

$$\therefore \dot{\psi} = \frac{1}{2} u C_{Lo} \cos\alpha_o \sin\phi \sec\theta = (\bar{k}/2r_o) C_{L0} \cos\alpha_0 \sin^2\phi. \tag{16}$$

From equations (10) and (14),

$$\frac{1}{r_0}\frac{d}{d\tau}(\cos\theta\sin\phi) = -\sin\theta. \tag{17}$$

From equations (8) and (12),

$$(\dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi)\, u = \frac{1}{2} u^2 C_{L0}\cos\alpha_0 - \bar{k}\cos\theta\cos\phi,$$

and substituting for $\dot{\psi}$ and u from equations (9) and (14) respectively,

$$\dot{\theta} = r_0 \left(\frac{\bar{k} C_{L0}}{2 r_0^2}\cos\alpha_0\cos\theta\sin^2\phi\cos\phi - 1\right)\operatorname{cosec}\phi. \tag{18}$$

Then, from equations (17) and (18),

$$\frac{1}{r_0}\frac{d}{d\theta}(\cos\theta\sin\phi) = \frac{-\sin\theta\sin\phi}{r_0\left(\frac{\bar{k} C_{L0}}{2r_0^2}\cos\alpha_0\cos\theta\sin^2\phi\cos\phi - 1\right)}$$

$$\therefore \cos\theta\cos\phi\frac{d\phi}{d\theta}\left(1 - \frac{\bar{k} C_{L0}}{2r_0^2}\cos\alpha_0\cos\theta\sin^2\phi\cos\phi\right)$$

$$= \sin\theta\sin\phi\left(2 - \frac{\bar{k} C_{L0}}{2r_0^2}\cos\alpha_0\cos\theta\sin^2\phi\cos\phi\right). \tag{19}$$

In order to obtain a simple analytical solution of this equation it is assumed that

$$\frac{\bar{k} C_{L0}}{2r_0^2} \cos \alpha_0 \cos \theta \sin^2 \phi \cos \phi \ll 1. \tag{20}$$

Then equation (19) becomes

$$\cos \theta \cos \phi \frac{d\phi}{d\theta} = 2 \sin \theta \sin \phi,$$

or

$$\sin \phi = b/\cos^2\theta,$$

where

$$b = \sin \phi_0 .$$

Now from equation (14),

$$u = \bar{k} b / r_0 \cos \theta$$

and from equation (16),

$$\dot{\psi} = \bar{k}\, b^2 C_{L0} \cos \alpha_0 / 2r_0 \cos^4\theta.$$

Finally, from equation (18),

$$\dot{\theta} = - r_0 \operatorname{cosec} \phi = - r_0 \cos^2\theta / b.$$

$$\therefore \tan \theta = - r_0 \tau / b,$$

showing that the attitude angle θ becomes more negative as time progresses.

The remaining variables may now be expressed as functions of time, and the main features of the incipient slow spin are demonstrated. The theory breaks down when $-\theta$ becomes too large, since the assumption (20) is equivalent to $\cos \theta \approx 1$.

The spin is slow, since r_0 is of order zero. Taking $r_0 = 1$, the number of turns per minute is

$$\frac{60}{2\pi\hat{t}} = \frac{60 V_0'}{2\pi\mu s} \approx \frac{60}{s},$$

assuming $V_0' \approx 100$ f.p.s. and $\mu = 16$ (of order -2).

Since a typical small aeroplane has a wing semi-span s of 15 or 20 feet, the theory predicts a spin of three or four turns per minute, as stated by Brodetsky.

4. *The Fast Spin*

Results from full-scale spinning trials [3], [4], [5] show that in an oscillatory spin the angular velocity in pitch is usually much smaller than the angular velocities in roll and yaw. We therefore consider solutions of equations (1) - (9) for which

$$0(p) = -1, \ 0(q) = 0, \ 0(r) = -1,$$

again taking C_{D0} to represent first order smallness.
We also assume

$$0(C') = 0, \ 0(\mu) = -3, \ 0(m_q/i_B) = -2, \ 0(n_v/i_c) = 1,$$

$$0(v) = 0(w) = 0(\delta w) = 0,$$

while the magnitudes of all other quantities remain unchanged from Section 3. The change in $0(C')$ and $0(\mu)$ are made so that these quantities are now more representative of modern high-speed aircraft, which have nearly all their mass concentrated in the fuselage, have a high wing loading, and fly at high altitudes. Also, it is logical to assume in the present case that the aerodynamic derivative m_q, representing damping in pitch, is large, since the angular velocity in pitch has been assumed small. Sometimes it may be necessary to include terms involving second derivatives, such as l_{vw}, and also terms involving acceleration derivatives, such as $n_{\dot{v}}$. In the present theory these terms are neglected, though they should be carefully investigated for any given aircraft configuration and conditions of flight - an account of all the relevant derivatives is given by Thomas [6].

A first approximation to the motion is found by retaining only the terms of greatest magnitude in each of the equations (1) - (6). This gives

$$\dot{u} - rv = 0, \tag{21}$$

$$\dot{v} - pw + ru = 0, \tag{22}$$

$$\dot{w} + pv = 0, \tag{23}$$

$$\dot{p} = v'\mu l_v v/i_A, \tag{24}$$

$$\dot{q} - B'rp = v'(\mu m_w \delta w + m_q q)/i_B, \tag{25}$$

$$\dot{r} = v'\mu n_v v/i_c. \tag{26}$$

The kinematic equations (7) - (9) are not considered further, since they are now independent of equations (21) - (26), owing to the relative smallness of the gravitational terms. It is interesting to note that the gravitational terms are also negligible in a first approximation to the fast rolling motion of an aircraft in the case when inertia cross-coupling produces large disturbances in incidence and sideslip: see reference [7].

An exact solution of equations (21) - (26) is now obtained. From equations (21) - (23) we obtain

$$v'^2 \equiv u^2 + v^2 + w^2 = \text{const.},$$

and therefore

$$v' = v'_0 = 1.$$

The angular velocity in pitch q appears only in equation (25); this equation is therefore used to find q after the remaining equations have been solved.

Introducing the notation

$$l'_v = \mu l_v / i_A, \ n'_v = \mu n_v / i_c,$$

equations (24) and (26) give

$$p - p_0 = (l'_v/n'_v)\ (r - r_0). \tag{27}$$

From equations (23) and (26), and using equation (27),

$$\delta w = \frac{(-n'_v p_0 + l'_v r_0)}{n'^2_v}\ (r - r_0) - \frac{l'_v}{2n'^2_v}\ (r^2 - r_0^2). \tag{28}$$

From equations (21) and (26)

$$u - u_0 = (r^2 - r_0^2)/2n'_v. \tag{29}$$

Then equation (22) becomes

$$\dot{v} = Ar^3 + Br^2 + Cr + D, \tag{30}$$

where

$$A = -\frac{l'^2_v}{2n'^3_v} - \frac{1}{2n'_v},$$

$$B = -\frac{3l'_v}{2n'^2_v}\left(p_0 - \frac{l'_v r_0}{n'_v}\right),$$

$$C = \frac{l'_v}{n'_v}\left[w_0 - \frac{(l'_v r_0 - n'_v p_0)}{n'^2_v}\ r_0 + \frac{l'_v r_0^2}{2n'^2_v}\right] - \frac{(n'_v p_0 - l'_v r_0)^2}{n'^3_v} - u_0 + \frac{r_0^2}{2n'_v},$$

$$D = \left(p_0 - \frac{l'_v r_0}{n'_v}\right)\left[w - \frac{(l'_v r_0 - n'_v p_0)}{n'^2_v} + \frac{l'_v r_0^2}{2n'^2_v}\right].$$

Since

$$\ddot{v} = \ddot{r}/n'_v,$$

integration of equation (30) gives

$$\dot{r}^2 = ar^4 + br^3 + cr^2 + dr + e \equiv w_1, \text{ say,} \tag{31}$$

where

$$a = \frac{1}{2}An'_v, \quad b = \frac{2}{3}Bn'_v, \quad c = Cn'_v, \quad d = 2Dn'_v,$$

$$e = \dot{r}_0^2 - 2n'_v\left(\frac{1}{4}Ar_0^4 + \frac{1}{3}Br_0^3 + \frac{1}{2}Cr_0^2 + Dr_0\right),$$

and A, B, C, D are the coefficients in equation (30).

Integration of equation (31) gives

$$\tau = \operatorname{sgn}(\dot{r})\int \frac{dr}{w_1^{\frac{1}{2}}},$$

which is an elliptic integral of the first kind. Reducing this integral to the standard from (see Appendix 1) we find in a particular case that

$$r = \frac{\alpha + \beta\, R\, \mathrm{cn}\,(G\tau + h)}{1 + R\, \mathrm{cn}\,(G\tau + h)}, \tag{32}$$

where the constants α, β, R, G and h are defined in Appendix 1. Hence r oscillates between the values

$$\frac{\alpha + \beta R}{1 + R} \text{ and } \frac{\alpha - \beta R}{1 - R},$$

the time of a complete period being $4K/G$.

Expressions for the remaining variables are easily obtained. On differentiating equation (32), we find

$$v = \frac{\dot{r}}{n_v} = \frac{(\alpha - \beta) RG\, \mathrm{sn}\, T\, \mathrm{dn}\, T,}{n'_v (1 + R\, \mathrm{cn}\, T)^2} \tag{33}$$

where

$$T = G\tau + h,$$

while p, δw and u are obtained from equations (27), (28) and (29) respectively. Finally, q is obtained by integration of equation (25), giving

$$q = e^{m'_q \tau} \int_0^{\tau} e^{-m'_q s} (B' r p + m'_w \delta_w)\, ds, \tag{34}$$

where

$$m'_q = m_q / i_B, \quad m'_w = \mu m_w / i_B,$$

r, p and δw are known functions of τ, and it is assumed that $q=0$ when $\tau=0$. The integration in equation (34) is performed numerically.

All the variables with the exception of q are seen to be periodic, while q has the form of a damped forced oscillation; we have therefore obtained an approximation to an oscillatory spin. A numerical example of the theory of this Section is given in Appendix 2.

5. *Conclusions*

After showing how Brodetsky's theory leads to a description of the incipient slow spin, we have obtained, by similar methods, a first approximation to an oscillatory spin in the practical case when q is small compared with p and r. The results show the expected non-linear oscillations in the velocities and angular velocities.

A consideration of orders of magnitude indicates that gravitational terms appear in a *second* approximation to a fast spin; in Brodetsky's theory of the slow spin they appear in a *first* approximation. The equations of motion representing a second approximation to a fast spin have no analytic solutions, so that investigations with the aid of a computer are indicated.

Solutions by computer of the complete equations of motion in which the derivatives are allowed to vary with incidence often show random and confusing motions after the initial stall - such motions are not necessarily incipient spins [2]. Thus extreme care is needed when interpreting the results of approximate analytic methods applied to spinning problems. However, the results of the present paper indicate that valuable information on at least some aspects of the spin may be obtained by such methods.

APPENDIX 1

DERIVATION OF EQUATION (32)

Referring to equation (31), let

$$ar^4 + br^3 + cr^2 + dr + e = (r^2 + 2B_1r + C_1)(ar^2 + 2B_2r + C_2),$$

with B_1, B_2, C_1, C_2 all real, and where, if all the roots of $w_1 = 0$ are real, the roots of the quadratics do not interlace. Let λ_1, λ_2 be the roots of the quadratic

$$(B_2^2 - aC_2)\lambda^2 + (aC_1 - 2B_1B_2 + C_2)\lambda + (B_1^2 - C_1) = 0. \tag{35}$$

Making the substitution

$$t = (r - \alpha)/(r - \beta), \tag{36}$$

where

$$\alpha, \beta = -\frac{(B - \lambda_{1,2}B_2)}{1 - a\lambda_{1,2}},$$

it can be shown that [8]

$$\int \frac{dr}{w_1^{\frac{1}{2}}} = \frac{|\lambda_2 - \lambda_1|}{(\alpha - \beta)\,|1 - a\lambda_1|} \int \frac{dt}{\left\{\left[t^2 - \left(\dfrac{1-a\lambda_2}{1-a\lambda_1}\right)\right]\left[\lambda_2 t^2 - \dfrac{\lambda_1(1-a\lambda_2)}{1-a\lambda_1}\right]\right\}^{\frac{1}{2}}} \tag{37}$$

The subsequent substitutions depend on the values of λ_1 and λ_2. To take a definite case, we assume

$$\lambda_1 > 0,\ \lambda_2 < 0,\ 1 - a\lambda_2 < 0.$$

The integral on the right-hand side of equation (37) is then

$$(-\lambda_2)^{\frac{1}{2}} \int \frac{dt}{\left[(R^2 - t^2)(t^2 + S^2)\right]^{\frac{1}{2}}}$$

where

$$R^2 = \frac{\lambda_1(1-a\lambda_2)}{\lambda_2(1-a\lambda_1)}, \quad S^2 = -\frac{1-a\lambda_2}{1-a\lambda_1}.$$

Now let

$$t = -R\,(1-\mu)^{\frac{1}{2}}, \quad R > 0. \tag{38}$$

Then

$$\int \frac{dt}{\left[(R^2 - t^2)(t^2 + S^2)\right]^{\frac{1}{2}}} = \frac{1}{(R^2+S^2)^{\frac{1}{2}}} \int \frac{du}{\left[(1-u^2)(1-k^2u^2)\right]^{\frac{1}{2}}}$$

$$= \frac{\mathrm{sn}^{-1} u}{(R^2+S^2)^{\frac{1}{2}}} + \text{const.}$$

where

$$k^2 = R^2/(R^2+S^2).$$

Therefore

$$\tau = \mathrm{sgn}\,(\dot{r}) \int \frac{dr}{w_1^{\frac{1}{2}}} = \mathrm{sgn}\,(\dot{r}) \frac{(\lambda_1 - \lambda_2)\,\mathrm{sn}^{-1} u}{(\alpha-\beta)(1-a\lambda_1)(-\lambda_2)^{\frac{1}{2}}(R^2+S^2)^{\frac{1}{2}}} + \text{const.},$$

i.e.

$$u = \mathrm{sn}\,(G\tau + h), \tag{39}$$

where

$$G = \frac{(\alpha-\beta)(1-a\lambda_1)(-\lambda_2)^{\frac{1}{2}}(R^2+S^2)^{\frac{1}{2}}}{\lambda_1 - \lambda_2}$$

and h is a constant of integration.

Equation (32) now follows immediately from equations (36), (38) and (39).

APPENDIX 2

NUMERICAL EXAMPLE BASED ON SECTION 4

The theory of Section 4 is illustrated by taking

$$B' = 0.9, \quad l'_v = -30, \quad m'_w = -45, \quad n'_v = 45, \quad m'_q = -10,$$

with initial conditions

$$p_o = 5.0, \quad q_o = 0, \quad r_o = 5.0,$$

$$u_o = 0.8, \quad v_o = 0.4, \quad w_o = 0.447.$$

Note that

$$u_o^2 + v_o^2 + w_o^2 = 1.$$

The coefficients in equations (30) and (31) are

$$A = -0.01605,\ B = 0.1852,\ C = -2.8572,\ D = 9.8975,$$

$$a = -0.3611,\ b = 5.556,\ c = -128.6,\ d = 890.8,\ e = -1384.$$

In the notation of Appendix 1,

$$B_1 = -4.1785,\ C_1 = 13.43,\ B_2 = 1.2698,\ C_2 = -102.52.$$

The roots of the quadratic equation (35) are

$$\lambda_1 = 0.04104,\ \lambda_2 = -2.7737.$$

Hence

$$\alpha = 4.1689,\ \beta = -414.69,\ R^2 = 2.308 \times 10^{-5},$$

$$S^2 = 1.560 \times 10^{-3},\ k = 0.1207,\ G = 10.005.$$

Inserting the initial conditions in equation (32), we find

$$\text{cn } h = \frac{\alpha - r_o}{R(r - \beta)} = -0.4122,$$

and since $v_o > 0$, equation (33) shows that sn h > 0. Hence

$$\text{sn } h = 0.9111,\ h = 2.0045.$$

Then, from equation (32),

$$r = \frac{4.169 - 1.992 \text{ cn } (10.005\ \tau + 2.0045)}{1 + 0.004804 \text{ cn } (10.005\ \tau + 2.0045)}.$$

Hence r oscillates between the values 2.167 and 6.191. A check on the computation is provided by the fact that these values are the real roots of

$$\dot{r}^2 (r) = 0,$$

i.e. of

$$r^2 + 2 B_1 r + C_1 = 0.$$

The periodic time of r is

$$4K/G = 0.630 \text{ airsecs.} = 0.630\ \hat{t} \text{ secs.}$$

The remaining variables, apart from q, which is obtained numerically from equation (34), are given by

$$u = 0.01111\ r^2 + 0.5222,$$

$$v = \frac{0.4474 \text{ sn } T \text{ dn } T}{(1 + 0.004804 \text{ cn } T)^2}.$$

$$\delta w = 0.007407\ r^2 - 0.1852\ r + 0.7407,$$

$$p = 8.333 - 0.667\ r,$$

where

$$T = 10.005\,\tau + 2.0045.$$

The results are tabulated below for $0 \leqslant \tau \leqslant 0.60$. The computation can, of course, be extended beyond this time as long as the variables do not exceed their assumed orders of magnitude.

τ (airsecs)	p	q	r	u	v	w
0	5	0	5	0.8	0.4	0.447
0.05	4.48	1.02	5.78	0.89	0.27	0.37
0.10	4.22	1.71	6.17	0.95	0.07	0.33
0.15	4.29	2.18	6.07	0.93	-0.16	0.34
0.20	4.67	2.38	5.50	0.86	-0.34	0.39
0.25	5.26	2.72	4.61	0.76	-0.43	0.49
0.30	5.92	2.03	3.62	0.67	-0.43	0.61
0.35	6.49	1.66	2.77	0.61	-0.32	0.73
0.40	6.83	0.92	2.26	0.58	-0.13	0.81
0.45	6.86	0.48	2.21	0.58	0.09	0.82
0.50	6.58	0.24	2.64	0.60	0.29	0.75
0.55	6.04	0.41	3.44	0.65	0.42	0.64
0.60	5.39	0.86	4.42	0.74	0.44	0.51

NOTATION

a, b, c, d, e — coefficients in equation (31)

$A' = (B-C)/A$, $B' = (C-A)/B$, $C' = (A-B)/C$ — where A, B, C are the principal moments of inertia at the centre of gravity

A, B, C, D — coefficients in equation (30)

$b = \sin\phi_0$ — (Section 3)

B_1, C_1, B_2, C_2 — constants defined in Appendix 1

C_D — total aircraft drag coefficient

C_L — total aircraft lift coefficient

G — constant defined in equation (39)

h — constant of integration

i_A = A/ms^2, where A is the longitudinal principal moment of inertia at the centre of gravity. Similarly for i_B, i_C.

k = $R/(R^2 + S^2)^{\frac{1}{2}}$, modulus of elliptic functions (Appendix 1)

$\bar{k}$ — $mg/\rho S V_0'^2$

K — complete elliptic integral of the first kind, with modulus k.

m — mass of aircraft

0 — order of magnitude

p, q, r		non-dimensional components of angular velocity: $p = P\hat{t}$, etc., where P, Q, R are the corresponding dimensional quantities
R, S		constants defined in Appendix 1
s		wing semi-span
S		wing area
$\text{sgn}(\dot{r})$	=	± 1 according as $\dot{r} \gtrless 0$
t		auxiliary variable defined in Appendix 1
$\hat{t}$	=	$m/\rho S V'_o$
T	=	$G\tau + h$
u, v, w		non-dimensional components of velocity: $u = U/V'_0$, etc., where U, V, W are the corresponding dimensional quantities
u		auxiliary variable defined in Appendix 1
v'	=	$(u^2 + v^2 + w^2)^{\frac{1}{2}}$, non-dimensional resultant speed
V'	=	$(U^2 + V^2 + W^2)^{\frac{1}{2}}$, dimensional resultant speed
w_1		defined by equation (31)
α		incidence of longitudinal principal axis
α, β		constants defined in Appendix 1
γ		gliding angle
δw		increment in w
θ		attitude angle
λ_1, λ_2		roots of equation (35), Appendix 1
μ	=	$m/\rho Ss$, relative density parameter
ρ		air density
τ		time in airsecs. (time in seconds is $\tau\hat{t}$)
ϕ		angle of bank
ψ		azimuth angle

Suffix 0 denotes the initial value of a variable. Dots denote differentiation with respect to τ.

Aerodynamic stability derivatives

The non-dimensional stability derivatives are referred to the principal axes at the centre of gravity, and are defined in terms of the corresponding dimensional stability derivatives as follows:

$X_w = \rho S V'_0 x_w$	for force-velocity derivatives
$L_v = \rho S V'_0 s l_v$	for moment-velocity derivatives
$L_r = \rho S V'_0 s^2 l_r$	for moment-angular velocity derivatives
$M_{\dot{w}} = \rho S s^2 m_{\dot{w}}$	for moment-acceleration derivatives
$L_{vw} = \rho S s\, l_{vw}$	

Also

$l'_v = \mu l_v / i_A$

$m'_w = \mu m_w / i_B, \quad m'_q = m_q / i_B$

$n' = \mu n \;/i$

References

1. Brodetsky, S.: The general motion of the aeroplane. Phil. Trans. Roy. Soc. Series A, 238, pp. 305-355, 1940.

2. Scher, S.H. et al: Analytical investigation of effect of spin entry technique on spin and recovery characteristics for a 60° delta-wing airplane. N.A.S.A. T.N. D-156, 1959.

3. Kerr. T.H. and Dennis, D.R.: A comparison of the model and full scale spinning tests on a conventional straight wing aircraft (Balliol Mk. 2) with special reference to the oscillatory nature of the spin, R.A.E. Report Aero. 2480 (also A.R.C. Current Paper 15971), 1952.

4. Kerr, T.H.: An investigation of the spin and recovery characteristics of a conventional straight wing aircraft (Meteor Mk. 8) covering a wide range of inertias in pitch and roll. R.A.E. Report Aero. 2509, 1954.

5. Merewether, H.C.H.: Erect and inverted spinning with particular reference to the Hunter, Journ. Roy. Aero. Soc. 69, pp. 835-845, 1965.

6. Thomas, H.H.B.M.: State of the art of estimation of derivatives. A.G.A.R.D. Report 339. N.A.T.O., 1961.

7. Walsh, G.R.: Forced autorotation in the rolling motion of an aeroplane. The Aeronautical Quarterly Vol. XVII, pp. 269-284, 1966.

8. Whittaker, E.T. and Watson, G.N. Modern Analysis. Cambridge University Press, 1950.

[Received May 2, 1967]